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ABSTRACT

An error analysis for mixed finite element approximations to the equations arising from periodic wave propagation is given. Special attention is given to the order of the convergence as well as the effect of the frequency on the errors.

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§1 Introduction.

Let $p, \underset{\sim}{u}, \rho$ denote the fluid pressure, velocity, and density. Our starting point is the Eulerian equations of motion

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} . \mathrm{grad})\underset{\sim}{u} + \frac{1}{\rho} \,\mathrm{grad}\, p = 0 \tag{1.1}$$

$$\frac{\partial p}{\partial t} + (\underset{\sim}{u} . \mathrm{grad}) p + \gamma p \,\mathrm{div}\, \underset{\sim}{u} = 0 \tag{1.2}$$

$$p\rho^{-\gamma} = \text{constant} \quad , \tag{1.3}$$

where γ is a ratio of specific heats. The particular case of interest in this paper is the acoustic disturbance about a mean flow $p_o, \underset{\sim}{u}_o, \rho_o$, where the pressure has the form

$$p = p_o + e^{irt} p_1 \quad , \quad |p_1/p_o| << 1. \tag{1.4}$$

Assuming for example the mean flow is uniform and neglecting quadratic terms we obtain the following:

$$ir\underset{\sim}{u} + \frac{1}{\rho_o} \mathrm{grad}\, p = 0 \tag{1.5}$$

$$\gamma p_o \mathrm{div}\, \underset{\sim}{u} + irp = 0 \tag{1.6}$$

These equations are required to hold in the flow region Ω with linear combinations of the pressure and normal velocity being specified on the boundary Γ; say

$$\alpha \, \underset{\sim}{u} \cdot \underset{\sim}{\nu} + \beta p = \sigma \quad \text{on} \quad \Gamma, \tag{1.7}$$

where $\underset{\sim}{\nu}$ denotes the outer normal.

The goal of this paper is to analyze Galerkin or mixed variational approximations to the first order system (1.5)-(1.6).

At first glance such an approach may seem to be inferior to a discretization of (1.5)-(1.7) based on a least squares principle [1]. The primary reason for this is that in the least squares approach one can obtain second order accuracy in L_2 for both $\underset{\sim}{u}$ and p by using appropriate piecewise linear spaces for each. Such a combination, however, will be unstable in the Galerkin formulation ([2]-[4]). In the latter one typically uses an appropriate piecewise linear space for $\underset{\sim}{u}$ and a piecewise constant space for p. The degrees of freedom in the latter are virtually the same as in the least squares formulation for a comparable grid, yet yield only first order L_2 accuracy in p.

Two somewhat surprising results of the analysis in this paper offer hope, however, for the utility of the Galerkin approach. First we show that under suitable conditions one has superconvergence in the approximation to the pressure. For example, if $\hat{p}_h$ denotes the best L_2 approximation to p in a suitable space of piecewise constant functions, then we show that the L_2 error $\|p_h - \hat{p}_h\|_o$ in the Galerkin approximation p_h is actually second order if appropriate linear elements are used for the velocity $\underset{\sim}{u}$.

Second and of equal importance, the errors in the Galerkin approach do not deteriorate as rapidly when the frequency r increases. This is of particular importance in underwater acoustics where

$$r >> 1$$

is quite common [5].

This second property is also shared by the standard finite element formulation where (1.5) is differentiated and combined with (1.6) to give the Helmholtz equation

$$\Delta p + \omega p = 0, \tag{1.8}$$

and the Galerkin method is applied directly to (1.8). So long as the coefficients $p_o, \underset{\sim}{u}_o, \rho_o$ from the mean flow are smooth this approach is possibly preferable; however in many applications these coefficients come from measurements and are not smooth enough to be differentiated [1]. In such cases one must deal directly with a first order system like (1.5)-(1.6).

Previous work on Galerkin approximations has centered on the Poisson equation

$$\underset{\sim}{u} - \operatorname{grad} \varphi = 0 \tag{1.9}$$

$$\operatorname{div} \underset{\sim}{u} = f. \tag{1.10}$$

Let $\mathcal{V}^h$ denote the finite dimensional space of velocities and $\mathcal{S}^h$ the finite dimensional space of pressures. Brezzi [3] showed that optimal convergence in the norms

$$\|\psi\|_{\mathcal{S}} = \|\psi\|_o \qquad \|\underset{\sim}{v}\|_{\mathcal{V}} = \{\|\underset{\sim}{v}\|_o^2 + \|\operatorname{div} \underset{\sim}{v}\|_o^2\}^{1/2\,(*)} \tag{1.11}$$

will occur provided

$$\sup \left\{\frac{\int \psi \operatorname{div} \underset{\sim}{v}}{\|v\|_{\mathcal{V}}}\right\} \geqq \alpha \|\psi\|_{\mathcal{S}} \quad \text{all} \quad \psi \in \mathcal{S}^h \tag{1.12}$$

holds for a fixed $\alpha, 0<\alpha<\infty$, independent of h, where the sup is taken over all $\underset{\sim}{v}$ in $\mathcal{V}^h$. There are a variety of spaces satisfying this condition including piecewise linear functions on a suitable grid for the velocities and suitable piecewise constants for the pressure [6]. This combination gives first order accuracy in the mesh spacing h in the norms (1.11). These results are

alized in Section 3 for the acoustic equations.

In subsequent work ([2],[4]) on the Poisson equation it was shown that optimal accuracy in the norms

$$\|\psi\|_0 \ , \ \|\underset{\sim}{v}\|_0 \tag{1.13}$$

can be obtained under appropriate conditions. In particular, one needs the <u>inclusion property</u>

$$S^h = \operatorname{div}[V^h] \tag{1.14}$$

as well as a <u>decomposition property</u>. The latter states there is an α, $0<\alpha<\infty$, independent of h such that each $\underset{\sim}{v}_h \in V^h$ can be written

$$\underset{\sim}{v}_h = \underset{\sim}{w}_h + \underset{\sim}{z}_h , \tag{1.15}$$

where $\underset{\sim}{w}_h, \underset{\sim}{z}_h$ in V^h satisfy

$$\operatorname{div} \underset{\sim}{z}_h = 0 \ , \ \alpha\|\underset{\sim}{w}_h\|_0 \leq \|\operatorname{div} \underset{\sim}{v}_h\|_{-1} . \tag{1.16}$$

In many mixed finite element formulations the analogs of (1.14)-(1.16) and (1.12) are equivalent ([7]). In this setting they are not. In fact, one can show that in this case (1.12) is equivalent to an inclusion and decomposition property but with different norms in (1.16) [7]. Nevertheless the finite element spaces which are known to satisfy (1.14)-(1.16) also satisfy (1.12) and conversely ([4]). In Section 4 we generalize the error estimates using (1.14)-(1.16) to cover the acoustic equations. The most important aspect of this analysis is the superconvergence in the pressure p.

§2 The Galerkin formulation. For simplicity we consider the boundary value problem

$$\underset{\sim}{u} - \operatorname{grad} \varphi = \underset{\sim}{f} \quad \text{in} \quad \Omega \tag{2.1}$$

$$\operatorname{div} \underset{\sim}{u} + \omega\varphi = g \quad \text{in} \quad \Omega \tag{2.2}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma \tag{2.3}$$

The mixed variational formulation of this problem is based on Galerkin's method and takes the following form. Given $\underset{\sim}{f}_0 \in L_2(\Omega)$, $g_0 \in L_2(\Omega)$ find

$$\underset{\sim}{u}_0 \in V = H(\operatorname{div};\Omega)^*, \quad \varphi_0 \in S = L^2(\Omega) \tag{2.4}$$

such that

$$a(\underset{\sim}{u}_0, \underset{\sim}{v}) + b(\underset{\sim}{v}, \varphi_0) = (\underset{\sim}{f}_0, \underset{\sim}{v}) \tag{2.5}$$

$$b(u_0, \psi) + c(\varphi_0, \psi) = \langle g_0, \psi \rangle \tag{2.6}$$

holds for all $v \in V$, $\psi \in S$. The forms are defined as follows:

$$a(\underset{\sim}{u}, \underset{\sim}{v}) = (\underset{\sim}{u}, \underset{\sim}{v}) = \int_\Omega \underset{\sim}{u} \cdot \underset{\sim}{v}, \quad b(\underset{\sim}{v}, \psi) = \int_\Omega \psi \operatorname{div} \underset{\sim}{v} \tag{2.7}$$

$$c(\varphi, \psi) = \int_\Omega \omega\varphi\psi \qquad \langle \varphi, \psi \rangle = \int_\Omega \varphi\psi \tag{2.8}$$

(*) $H(\operatorname{div};\Omega)$ consists of $\underset{\sim}{v} \in \vec{L}_2(\Omega)$ such that $\operatorname{div} \underset{\sim}{v} \in L_2(\Omega)$.

To approximate we introduce finite dimensional spaces

$$\mathcal{S}^h \subseteq \mathcal{S} \ , \quad \mathcal{V}^h \subset \mathcal{V} \tag{2.9}$$

and seek $\underset{\sim}{u}_h \in \mathcal{V}^h$, $\psi_h \in \mathcal{S}^h$ such that (2.5)-(2.6) holds (with $\underset{\sim}{u}_0$ replaced with $\underset{\sim}{u}_h$ and φ_0 replaced with φ_h) for all $\underset{\sim}{v} \in \mathcal{V}^h$, $\psi \in \mathcal{S}^h$.

It may happen that (2.1)-(2.3) does not have a unique solution, a case which arises for example if ω is an eigenvalue of the homogeneous problem. We explicitly rule this out by assuming that the adjoint equation (which in this case is the same as (2.1)-(2.3)) is uniquely solvable. More precisely, we assume that for each pair $\hat{\underset{\sim}{f}} \in H(\mathrm{div},\Omega)$, $\hat{g} \in L_2(\Omega)$ there us a unique pair $\hat{\underset{\sim}{v}} \in \mathcal{V}$, $\hat{\psi} \in \mathcal{S}$ for which

$$a(\underset{\sim}{w},\hat{\underset{\sim}{v}}) + b(\underset{\sim}{w},\hat{\psi}) = (\hat{\underset{\sim}{f}},\underset{\sim}{w}) \tag{2.11}$$

$$b(\hat{\underset{\sim}{v}},\xi) + c(\xi,\hat{\psi}) = \langle \hat{g},\psi \rangle \tag{2.12}$$

holds for all $\underset{\sim}{w} \in \mathcal{V}$, $\xi \in \mathcal{S}$. Moreover, we assume that the solution of (2.11)-(2.12) satisfies the standard apriori bound for the Helmholtz equation

$$\Delta\hat{\psi} + \omega\hat{\psi} = \hat{g} - \mathrm{div}\ \hat{f}.$$

Namely,

$$\|\hat{\underset{\sim}{v}}\|_0 + \|\hat{\psi}\|_1 \leqq K_0(\|\hat{g}\|_{-1} + \|\hat{\underset{\sim}{f}}\|_0) \tag{2.13}$$

The constant K_0 approaches infinity with $\sqrt{|\omega|}$.

The effect of the frequency ω in our analysis will also be seen in the constant $0 < K_1 < \infty$ satisfying

$$|c(\varphi,\psi)| \leq K_1 \|\varphi\|_0 \|\psi\|_0 \quad \text{all} \quad \varphi,\psi \in \mathcal{S} \tag{2.14}$$

For the model problem (2.1)-(2.3) we can take $K_1 = \omega$. In the general acoustic equations it will be a more complicated function of ω but will still approach infinity linearly with ω.

As noted in the introduction the case $\omega = 0$ has received considerable attention. We shall use this work to define a mapping from $\mathcal{V}\times\mathcal{S}$ to $\mathcal{V}^h\times\mathcal{S}^h$ - called the Poisson projector P^h -- as follows. We let

$$\{\hat{\underset{\sim}{v}}_h,\hat{\psi}_h\} = P^h\{\hat{\underset{\sim}{v}},\hat{\psi}\}$$

when

$$a(\underset{\sim}{w}^h,\hat{\underset{\sim}{v}}_h) + b(\underset{\sim}{w}^h,\psi_h) = a(\underset{\sim}{w}^h,\hat{\underset{\sim}{v}}) + b(\underset{\sim}{w},\hat{\psi}) \tag{2.15}$$

$$b(\hat{\underset{\sim}{v}}_h,\xi^h) = b(\hat{\underset{\sim}{v}},\xi^h) \tag{2.16}$$

holds for all $\underset{\sim}{w}^h\in\mathcal{V}^h$, $\xi^h\in\mathcal{S}^h$. That is $\{\hat{\underset{\sim}{v}}_h,\hat{\psi}_h\}$ is the mixed finite element approximation to the Poisson equation generated by $\{\hat{\underset{\sim}{v}},\hat{\underset{\sim}{\psi}}\}$.

If Brezzi's condition (1.12) holds, then there is a constant $0 < C_\alpha < \infty$, depending only on the number α in (1.12) such that

$$\|\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h\|_0 + \|\operatorname{div}(\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h)\|_0 + \|\hat{\psi}_0 - \hat{\psi}_h\|_0 \leq C_\alpha E_1(\hat{\underset{\sim}{v}},\hat{\psi}), \tag{2.17}$$

where

$$E_1(\hat{\underset{\sim}{v}},\hat{\psi}) = \inf\{\|\hat{\underset{\sim}{v}} - \underset{\sim}{w}^h\|_0 + \|\mathrm{div}(\hat{\underset{\sim}{v}} - w^h)\| + \|\hat{\psi} - \xi^h\|_0\} \qquad (2.18)$$

and where the inf is taken over $\underset{\sim}{w}^h \epsilon \mathcal{V}$, $\xi^h \epsilon \mathcal{S}^h$. (See [3])

Similarly if the inclusion and decomposition properties (1.14)-(1.16) hold, then

$$\|\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h\|_0 \leqq C_\alpha E_0(\hat{\underset{\sim}{v}}) \qquad (2.19)$$

and

$$\|\hat{\psi} - \hat{\psi}_h\|_0 \leqq C_\alpha[(E_0(\hat{\underset{\sim}{v}}_0) + E_0(\hat{\psi}_0)], \qquad (2.20)$$

where

$$E_0(\hat{\underset{\sim}{v}}_0) = \inf\|\hat{\underset{\sim}{v}} - \underset{\sim}{w}^h\|_0, \quad E_0(\hat{\psi}) = \inf\|\hat{\psi} - \xi^h\|_0. \qquad (2.21)$$

and now C_α depends on the constant α in (1.16) (See [4]).

Throughout this paper we shall assume that the spaces $\mathcal{V}^h$, $\mathcal{S}^h$ have the standard approximation properties. More precisely, we assume that we can approximate in $\mathcal{V}^h$ to order k in the sense that given $\underset{\sim}{w}\epsilon\vec{H}^k(\Omega)$

$$\inf\{\|\underset{\sim}{w} - \underset{\sim}{w}^h\|_0 + h\|\underset{\sim}{w} - \underset{\sim}{w}^h\|_1\} \leqq C_A h^k \|\underset{\sim}{w}\|_k \qquad (2.22)$$

for a fixed constant $0 < C_A < \infty$ independent of $\underset{\sim}{w}$ and h. Moreover, we assume (2.22) holds for k replaced with any smaller k_1, satisfying $1 < k_1 \leqq k$. Similarly, we assume that for any

$\xi \in H^{\ell}(\Omega)$

$$\inf \{\|\xi - \psi^h\|_0\} \leq C_A h^{\ell} \|\xi\|_{\ell} \tag{2.23}$$

with this inequality holding for ℓ replaced with any ℓ_1 satisfying $0 < \ell_1 \leq \ell$. For most spaces satisfying the Brezzi condition (1.12) or the inclusion condition (1.14) we have $\ell = k - 1$.

Observe that if (2.22)-(2.23) hold, then

$$E_1(\underset{\sim}{w}, \xi) = O(h^{k-1}) + O(h^{\ell})$$

while

$$E_0(\underset{\sim}{w}) = O(h^k), \qquad E_0(\xi) = O(h^{\ell})$$

provided $w \in H^k(\Omega)$, $\xi \in \vec{H}^{\ell}(\Omega)$.

<u>§3 The first error estimates</u>. In this section we assume that Brezzi's condition (1.12) is valid, and estimate the errors $\underset{\sim}{u}_0 - \underset{\sim}{u}_h$, $\varphi_0 - \varphi_h$. The proof that the discrete system (2.5)-(2.6) on $\mathcal{V}^h \times \mathcal{S}^h$ has a unique solution $\underset{\sim}{u}_h, \varphi_h$ is similar in structure to the error analysis so we shall give only the latter. Also we shall assume that the regularity (2.13) and approximability conditions (2.22)-(2.23) hold.

<u>Theorem 7</u>. Let (1.12) hold. Then there is a constant $0 < C < \infty$ depending only on α in (1.12) and ω such that if $hC < 1$, then

$$\|\underset{\sim}{u}_0 - \underset{\sim}{u}_h\|_0 + \|\varphi_0 - \varphi_h\|_0 \leq \left(\frac{C}{1-Ch}\right) E_1(\underset{\sim}{u}_0, \varphi_0) \tag{3.1}$$

Moreover, C approaches infinity linearly with $|\omega|^{3/2}$.

<u>Proof</u>. Let

$$B((\underset{\sim}{u},\varphi),(\underset{\sim}{v},\psi)) = a(\underset{\sim}{u},\underset{\sim}{v}) + b(\underset{\sim}{v},\varphi) + b(\underset{\sim}{u},\psi) + c(\varphi,\psi). \tag{3.2}$$

Then the defining equations (2.5)-(2.6) give

$$B((\underset{\sim}{u}_h,\varphi_h),(\underset{\sim}{v}^h,\psi^h)) = B((\underset{\sim}{u}_0,\varphi_0),(\underset{\sim}{v}^h,\psi^h)) \tag{3.3}$$

for all $\underset{\sim}{v}^h \in \mathcal{V}^h$, $\psi^h \in \mathcal{S}^h$. Let

$$\{\hat{\underset{\sim}{u}}_h, \hat{\varphi}_h\} \in \mathcal{V}^h \times \mathcal{S}^h$$

be given and put

$$\underset{\sim}{e}_h = \underset{\sim}{u}_h - \hat{\underset{\sim}{u}}_h, \qquad \underset{\sim}{e} = \underset{\sim}{u}_0 - \hat{\underset{\sim}{u}}_h \tag{3.4}$$

$$\epsilon_h = \varphi_h - \hat{\varphi}_h, \qquad \epsilon = \varphi_0 - \hat{\varphi}_h \tag{3.5}$$

Subtracting $B((\underset{\sim}{u}_h,\varphi_h),(\underset{\sim}{v}^h,\psi^h))$ from both sides of (3.3) gives

$$B((\underset{\sim}{e}_h,\epsilon_h),(\underset{\sim}{v}^h,\psi^h)) = B((\underset{\sim}{e},\epsilon),(\underset{\sim}{v}^h,\psi^h)) . \tag{3.5}$$

Let us first estimate the left hand side of (3.4), and in the process make a definite choice for $\underset{\sim}{v}^h$ and ψ^h. The idea is to choose these functions so that the left hand side becomes essentially $\|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2$. To do this we first solve the adjoint problem (2.11)-(2.12) with data $\underset{\sim}{f} = \underset{\sim}{e}_h, g = \epsilon_h$. Letting $\underset{\sim}{w} = \underset{\sim}{e}_h$, and $\xi = \epsilon_h$ in (2.11)-(2.12) gives

$$B((\underset{\sim}{e}_h,\epsilon_h),(\hat{\underset{\sim}{v}},\hat{\psi})) = \|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2. \tag{3.6}$$

Since $B(\cdot,\cdot)$ is linear in each variable it follows from (3.5) that

$$\|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2 = B((\underset{\sim}{e}_h,\epsilon_h),(\hat{\underset{\sim}{v}} - \underset{\sim}{v}^h,\hat{\psi} - \psi^h)) + B((\underset{\sim}{e},\epsilon),(\underset{\sim}{v}^h,\psi^h)) \tag{3.7}$$

We now let $\{\underset{\sim}{v}^h,\psi^h\} = \{\hat{\underset{\sim}{v}}_h,\hat{\psi}_h\}$ be the Poisson projection of $\{\hat{\underset{\sim}{v}}^h,\hat{\psi}\})$, i.e., (2.15)-(2.16) holds. Thus putting $\underset{\sim}{w}^h = \underset{\sim}{e}_h$, $\xi^h = \epsilon_h$ in (2.15)-(2.16) we get

$$a((\underset{\sim}{e}_h,\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h) + b(\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h,\epsilon_h) + b(\underset{\sim}{e}_h,\hat{\psi} - \hat{\psi}_h) = 0. \tag{3.8}$$

Thus

$$B((\underset{\sim}{e}_h,\epsilon_h),(\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h,\hat{\psi} - \hat{\psi}_h)) = c(\epsilon_h,\hat{\psi} - \hat{\psi}_h), \tag{3.9}$$

and so

$$\|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2 = c(\epsilon_h, \hat{\psi} - \hat{\psi}_h) + B((\underset{\sim}{e}, \epsilon), (\hat{v}_h, \hat{\psi}_h)). \tag{3.10}$$

We treat the second term on the right hand side of (3.10) in a similar way. In particular, we let $\{\hat{u}_h, \hat{\varphi}_h\}$ be the Poisson projection of $\{u_0, \varphi_0\}$. This gives

$$a(\underset{\sim}{e}, \hat{\underset{\sim}{v}}_h) + b(\underset{\sim}{e}, \hat{\psi}_h) + b(\hat{\underset{\sim}{v}}_h, \epsilon) = 0, \tag{3.11}$$

and so

$$B((\underset{\sim}{e}, \epsilon), (\hat{\underset{\sim}{v}}_h, \hat{\psi}_h)) = c(\epsilon, \hat{\psi}_h). \tag{3.12}$$

Combining this with (3.10) we obtain

$$\|\underset{\sim}{e}_h\|_0^2 + \|\epsilon_h\|_0^2 = c(\epsilon_h, \hat{\psi} - \hat{\psi}_h) + c(\epsilon, \hat{\psi}_h) \tag{3.13}$$

To estimate the first term on the right hand side of (3.13) we note that (2.14) gives

$$|c(\epsilon_h, \hat{\psi} - \hat{\psi}_h)| \leqq K_1 \|\epsilon_h\|_0 \|\hat{\psi} - \hat{\psi}_h\|_0. \tag{3.14}$$

Our approximation assumption (2.23) gives

$$\|\hat{\psi} - \hat{\psi}_h\|_0 \leqq C_A h \|\hat{\psi}\|_1 \tag{3.15}$$

The regularity (2.13) of (2.11)-(2.12) can be used to bound $\hat{\psi}$ in terms of the data $\underset{\sim}{e}_h, \epsilon_h$ as follows:

$$\|\hat{\psi}\|_1 \leqq K_0(\|\epsilon_h\|_0 + \|\underset{\sim}{e}_h\|_0) \tag{3.16}$$

Thus

$$|c(\epsilon_h, \hat{\psi} - \hat{\psi}_h)| \leq K_0 K_1 C_A h(\|\epsilon_h\|_0 + \|\underset{\sim}{e}_h\|_0)\|\epsilon_h\|_0 \qquad (3.17)$$

The second term on the right hand side of (3.13) is treated in a similar way. In particular,

$$\begin{aligned} |c(\epsilon, \psi_h)| &\leq K_1 \|\epsilon\|_0 (\|\hat{\psi} - \hat{\psi}_h\|_0 + \|\hat{\psi}\|_0) \\ &\leq K_1 \|\epsilon\|_0 (K_0 + C_A K_0 h)(\|\epsilon_h\|_0 + \|\underset{\sim}{e}_h\|_0) \qquad (3.18) \end{aligned}$$

Combining (3.17)-(3.18) with (3.13) we obtain (3.1).

<u>Remark</u>. The linear dependence on $|\omega|^{3/2}$ in the error estimates is an order of magnitude better than that obtained for the least squares approximation, where the dependence is quadratic.

The order of accuracy for a fixed ω is not best possible. For example, if the standard linear element - piecewise constant combination is used (i.e., $k = 2$ and $\ell = 1$), then we get only first order accuracy in $\underset{\sim}{u}_0$ and ω_0 as in [3] and [6].

§4 Improved estimates. In this section we assume that the inclusion and decomposition properties (1.14)-(1.16) hold, and show that the L_2 errors in $\underset{\sim}{u}_0$ are best possible. Moreover, we show that a superconvergence result holds for the scalar φ_0. The starting point is to prove the latter result for the Poisson projection, and then using the approach of the previous section show that it also holds for $\omega \neq 0$.

Lemma 1. Let $\{\hat{\underset{\sim}{v}}_h, \hat{\psi}_h\}$ be the Poisson projection of $\{\hat{\underset{\sim}{v}}, \hat{\psi}\}$ defined by (2.15)-(2.16), and let $\bar{\psi}_h$ be the best L_2 approximation to $\hat{\psi}$ in S^h. Then

$$\|\hat{\psi}_h - \bar{\psi}_h\|_0 \leq (C_\alpha/\alpha) E_0(\hat{\underset{\sim}{v}}) \tag{4.1}$$

Remark. If S^h consists of piecewise constant functions, then the value of $\bar{\psi}_h$ is a given triangle T is equal to the average of $\hat{\psi}$ over T. The above estimate states that this function will differ from the Poisson projection by order $O(h^2)$ if linear elements are used to represent $\underset{\sim}{u}_h$, or by order $O(h^3)$ if quadratic elements are used, to cite another popular combination.

Proof. Subtracting $\bar{\psi}^h$ in the right and left hand sides of (2.15) gives

$$b(\underset{\sim}{w}^h, \hat{\psi}_h - \bar{\psi}_h) = a(\underset{\sim}{w}^h, \hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h) + b(\underset{\sim}{w}^h, \hat{\psi} - \bar{\psi}_h) \tag{4.2}$$

for all $\underset{\sim}{w}^h \in V^h$. We use the inclusion property (1.14) to write

$$\hat{\psi}_h - \bar{\psi}_h = \operatorname{div} \underset{\sim}{y}_h \tag{4.3}$$

for $\underset{\sim}{y}_h \in \mathcal{V}^h$. In addition, we use the decomposition property (1.15)-(1.16) to write

$$\underset{\sim}{y}_h = \underset{\sim}{w}_h + \underset{\sim}{z}_h, \qquad \underset{\sim}{w}_h, \underset{\sim}{z}_h \in \mathcal{V}^h, \tag{4.4}$$

where

$$\operatorname{div} \underset{\sim}{z}_h = 0, \qquad \alpha \|\underset{\sim}{w}_h\|_0 \leqq \|\operatorname{div} \underset{\sim}{y}_h\|_{-1} \leqq \|\hat{\psi}_h - \bar{\psi}_h\|_0 \tag{4.5}$$

Let $\underset{\sim}{w}^h = \underset{\sim}{w}_h$ in (4.2). Then

$$b(\underset{\sim}{w}_h, \hat{\psi}_h - \bar{\psi}_h) = \int_\Omega \operatorname{div} \underset{\sim}{w}_h (\hat{\psi}_h - \bar{\psi}_h) = \int_\Omega (\hat{\psi}_h - \bar{\psi}_h)^2 \tag{4.6}$$

Also, since $\bar{\psi}_h$ is the best L_2 approximation

$$b(\underset{\sim}{w}_h, \hat{\psi} - \bar{\psi}_h) = \int_\Omega (\operatorname{div} \underset{\sim}{w}_h)(\hat{\psi} - \bar{\psi}) = 0 \tag{4.7}$$

(since $\operatorname{div} \underset{\sim}{w}_h \in \mathcal{S}^h$, by inclusion property). Thus, using (2.19) and (4.5), (4.2) becomes

$$\|\hat{\psi}_h - \bar{\psi}_h\|_0^2 = a(\underset{\sim}{w}_h, \hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h) \leqq \|\underset{\sim}{w}_h\|_0 \|\hat{\underset{\sim}{v}} - \hat{\underset{\sim}{v}}_h\|_0$$

$$\leqq \alpha^{-1} \|\hat{\psi}_h - \bar{\psi}_h\|_0 \, C_\alpha E_0(\hat{\underset{\sim}{v}}).$$

Cancellation of the common factor gives (4.1).

To apply this result to the case $\omega \neq 0$, we retain the approach of Section 3 except giving an alternate estimate for the term

$c(\epsilon,\psi_h)$. To treat the latter we note that

$$c(\epsilon,\hat{\psi}_h) = \int_\Omega \omega(\varphi_0 - \hat{\varphi}_h)\hat{\psi}_h, \tag{4.8}$$

where $\{\hat{\underset{\sim}{u}}_h,\hat{\varphi}_h\}$ is the Poisson projector of $\{\underset{\sim}{u}_0,\varphi_0\}$. Let $\bar{\varphi}_h$ denote the best L_2 approximation to φ_0 in S^h. Then

$$c(\epsilon,\hat{\psi}_h) = \int_\Omega \omega(\varphi_0 - \bar{\varphi}_h)\hat{\psi}_h + \int_\Omega \omega(\bar{\varphi}_h - \hat{\varphi}_h)\hat{\psi}_h \tag{4.9}$$

Assuming that ω is constant we have

$$\int_\Omega \omega(\varphi_0 - \bar{\varphi}_h)\hat{\psi}_h = 0, \tag{4.10}$$

since $\varphi_0 - \bar{\varphi}_h$ is orthogonal to $\underset{\sim}{S}^h$. Thus

$$|c(\epsilon,\hat{\psi}_h)| \leq K_1\|\bar{\varphi}_h - \bar{\varphi}\|_0\|\hat{\psi}_h\|_0 \tag{4.11}$$

Estimating the last term on the right of (4.11) as in (3.18) we obtain the following result.

<u>Theorem 2</u>. Let the inclusion and decomposition properties (1.14)-(1.16) and assume ω is a constant. There is a $0 < C < \infty$ depending only on α in (1.16) and ω such that if $Ch < 1$, then

$$\|\underset{\sim}{u}_0 - \underset{\sim}{u}_h\|_0 + \|\bar{\varphi}_h - \varphi_h\|_0 \leq \left(\frac{C}{1-Ch}\right)E_0(\underset{\sim}{u}_0), \tag{4.12}$$

where φ_h is the best L_2 approximation to φ_0 in S^h.

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